MATH4210: Financial Mathematics Tutorial 6

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Content Review

Question

(Distribution of Brownian motion)

$B_T - \cancel{B_0} \sim N(0,T)$ $\quad$ $=0$

$\Rightarrow \frac{B_T}{\sqrt{T}} \sim N(0,1)$

$$B_t - B_s \sim N(0, t-s)$$

$\text{Ex} = P(e^{B_T} > K) = P(B_T > \ln K) = P\left(\frac{B_T}{\sqrt{T}} > \frac{\ln K}{\sqrt{T}}\right) = 1 - P\left(\frac{B_T}{\sqrt{T}} \le \frac{\ln K}{\sqrt{T}}\right) = 1 - \Phi\left(\frac{\ln K}{\sqrt{T}}\right)$

$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ $\quad$ $N(0,1) \sim$

Question

$dS_t = \mu S_t dt + \sigma S_t dB_t$

$\Rightarrow$

S_t follows the Black Scholes Model with drift μ and volatility σ, we have

$$S_t = S_0 \exp((\mu - \sigma^2/2)t + \sigma B_t).$$

Under risk-neutral probability $\mathbb{Q}$, there exists a $\mathbb{Q}$-Brownian motion $B_t^{\mathbb{Q}}$ such that

$$S_t = S_0 \exp((r - \sigma^2/2)t + \sigma B_t^{\mathbb{Q}}).$$

$\Rightarrow S_T = S_t \exp\left((r - \frac{\sigma^2}{2})(T-t) + \sigma(B_T^{\mathbb{Q}} - B_t^{\mathbb{Q}})\right)$

interest rate

$S_T = S_0 \exp\left((r - \frac{\sigma^2}{2})T + \sigma B_T^{\mathbb{Q}}\right)$

Question

For an option with payoff function $g(S_T)$, the option price at time t is given by

$$u(t, S_t) := \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} g(S_T) | S_t],$$

interest rate

and $(e^{-rt} S_t)_{t\in[0,T]}$, $(e^{-rt} u(t, S_t))_{t\in[0,T]}$ are martingales.

$e^{-r(T-t)} f(B_T)$

~ BM

Question

$-ru$

If $u(t,x) := \mathbb{E}[f(B_T) | B_t = x]$, then $\partial_t u + \frac{1}{2}\partial^2_{xx} u = 0$.
If $u(t,x) := \mathbb{E}[f(X_T) | X_t = x]$, where $dX_t = \mu dt + \sigma dB_t$, then $\partial_t u + \frac{1}{2}\sigma^2 \partial^2_{xx} u + \mu \partial_x u = 0$.
If $u(t,x) := \mathbb{E}[f(S_T) | S_t = x]$, where $dS_t = \mu S_t dt + \sigma S_t dB_t$, then $\partial_t u + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} u + \mu x \partial_x u = 0$.

$-ru$

BS model

$e^{-r(T-t)} f(S_T)$

Question

(Computation of intergral)

(1) $\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx$

$N(0,1)$ $\int_R \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = 1 \Rightarrow \int_R e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$

odd

(2) $\int_{\mathbb{R}} x \cdot e^{-\frac{x^2}{2}}\, dx$ $=0$

$= \int_R e^{-\frac{x^2}{2}}\, d\left(\frac{x^2}{2}\right) = \int_R d\left(e^{-\frac{x^2}{2}}\right)\cdot(-1) = -e^{-\frac{x^2}{2}}\Big|_{-\infty}^{\infty} = 0$

(3) $\int_{\mathbb{R}} x^2 \cdot e^{-\frac{x^2}{2}}\, dx$ $= \int_R x \cdot e^{-\frac{x^2}{2}}\, d\left(\frac{x^2}{2}\right) = \int_R x \cdot d\left(e^{-\frac{x^2}{2}}\right)\cdot(-1) = \underbrace{(-1)\, e^{-\frac{x^2}{2}}\cdot x\Big|_{-\infty}^{\infty}}_{=0} + \int_R e^{-\frac{x^2}{2}}\, dx$ $= \sqrt{2\pi}$

(4) $\int_{\mathbb{R}} e^{-\frac{(x-a)^2}{2b}}\, dx$, here a, b are constants.

$y = \frac{x-a}{\sqrt{b}}$ $\quad \int_R e^{-\frac{y^2}{2}} \cdot \sqrt{b}\, dy$

$= \sqrt{b}\cdot\sqrt{2\pi} = \sqrt{2\pi b}$

(5) $\int_{\mathbb{R}} e^x \cdot e^{-\frac{x^2}{2}}\, dx$

$= \int_R e^{-\frac{x^2-2x}{2}}\, dx$

$= \int_R e^{-\frac{(x-1)^2-1}{2}}\, dx = e^{\frac{1}{2}} \underbrace{\int_R e^{-\frac{(x-1)^2}{2}}\, dx}_{\sqrt{2\pi}\cdot 1 \quad a=1\ b=1} = e^{\frac{1}{2}}\cdot\sqrt{2\pi}$

$(x^2 - 2x = (x-1)^2 - 1)$

Pricing by Martingale Approach

Question

Consider the stock price $(S_t)_{t\geq 0}$ which follows the Black Scholes model. Given risk-free interest rate r, find the price of a financial contract associated with S_t maturing at T with payoff function $g(x) := x^2$.

$g(S_T) = S_T^2$

Solution:
Since S_t follows the Black Scholes Model, we have

$$S_t = S_0 \exp((\mu - \sigma^2/2)t + \sigma B_t).$$

Under risk-neutral probability $\mathbb{Q}$, there exists a $\mathbb{Q}$-Brownian motion $B_t^{\mathbb{Q}}$ such that

$$S_t = S_0 \exp((r - \sigma^2/2)t + \sigma B_t^{\mathbb{Q}}).$$

Pricing by Martingale Approach

Denote by V_t the contract price at time $t \leq T$, the price of financial contract with payoff S_T^2. It is clear that $e^{-rt}V_t$ is a martingale under $\mathbb{Q}$. Notice that $V_T = g(S_T) = S_T^2$. Therefore,

$X_t = E(X_t|\mathcal{F}_s) = X_s$, $s \leq t$.

$$e^{-rt}V_t = \mathbb{E}^{\mathbb{Q}}[e^{-rT}V_T|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[e^{-rT}S_T^2|\mathcal{F}_t].$$

$= S_t^2 \exp\left((r-\frac{\sigma^2}{2})(T-t)+\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}})\right)^2$

$\sim N(0, T-t)$

$$
\begin{aligned}
V_t &= \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}S_T^2|\mathcal{F}_t] \\
&= S_t^2 e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\exp((r-\sigma^2/2)(T-t)+\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}}))^2|\mathcal{F}_t] \\
&= S_t^2 e^{-r(T-t)+(2r-\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}[\exp(2\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}}))|\mathcal{F}_t].
\end{aligned}
$$

Note that $B_T^{\mathbb{Q}} - B_t^{\mathbb{Q}}$ is independent of $\mathcal{F}_t$ with distribution $N(0, T-t)$ under $\mathbb{Q}$, then by characteristic function, we have

$= Y \sim N(0, T-t)$

$$\mathbb{E}^{\mathbb{Q}}[\exp(2\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}}))|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[e^{2\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}})}] = e^{2\sigma^2(T-t)}.$$

Finally, $V_t = S_t^2 e^{(r+\sigma^2)(T-t)}$. Taking $t=0$, we have $V_0 = S_0^2 e^{(r+\sigma^2)T}$.

① $X \sim N(\mu, \sigma^2) \Rightarrow E(e^X) = e^{\mu + \frac{\sigma^2}{2}}$

$E^Q(e^{2\sigma Y}) = e^{2\sigma^2(T-t)}$

$N(0, 4\sigma^2(T-t))$

② $E^Q[e^{2\sigma Y}] = E^Q[e^Z]$ $Z \sim N(0, 4\sigma^2(T-t))$

$= \int_R e^x \cdot \frac{1}{\sqrt{2\pi \cdot 4\sigma^2(T-t)}} \cdot e^{\frac{-x^2}{2\cdot 4\sigma^2(T-t)}} dx$

$= \frac{1}{\sqrt{2\pi \cdot 4\sigma^2(T-t)}} \int_R e^x \cdot e^{\frac{-x^2}{8\sigma^2(T-t)}} dx$

$= \frac{1}{\sqrt{2\pi \cdot 4\sigma^2(T-t)}} \int_R e^{-\frac{(x^2 - 8\sigma^2(T-t)x)}{8\sigma^2(T-t)}} dx$

$\left(\cancel{x}^2 - 8\sigma^2(T-t)x = (x - 4\sigma^2(T-t))^2 - (4\sigma^2(T-t))^2 \right)$

$(a+b)^2$

$= \frac{1}{\sqrt{8\sigma^2\pi(T-t)}} \int_R e^{-\frac{(x-4\sigma^2(T-t))^2 - (4\sigma^2(T-t))^2}{8\sigma^2(T-t)}} dx$

$= \cancel{\frac{1}{\sqrt{8\pi\sigma^2(T-t)}}} \cdot e^{2\sigma^2(T-t)} \int_R e^{-\frac{(x-4\sigma^2(T-t))^2}{8\sigma^2(T-t)}} dx$

$a = 4\sigma^2(T-t)$

$b = 4\sigma^2(T-t)$

$= \cancel{\sqrt{2\pi \cdot 4\sigma^2(T-t)}}$

$= e^{2\sigma^2(T-t)}$